Lecture 33

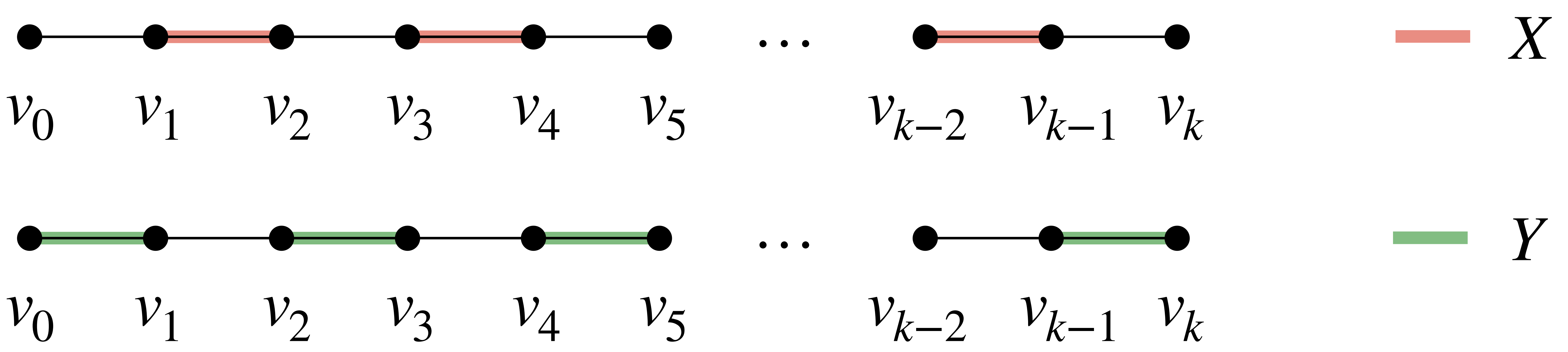
Matching (contd.), Hall's Marriage Theorem

Berge's Theorem

Theorem: A matching M is maximum if and only if there is no M-augmenting path.

Proof: ($\implies$) If there is an M-augmenting path, then M is not a maximum matching.

Let P be an M-augmenting path.



Let X be the set of edges in P that are in M and let Y be the rest of the edges.

Then $(M \backslash X) \cup Y$ will a be a larger matching than M.

…

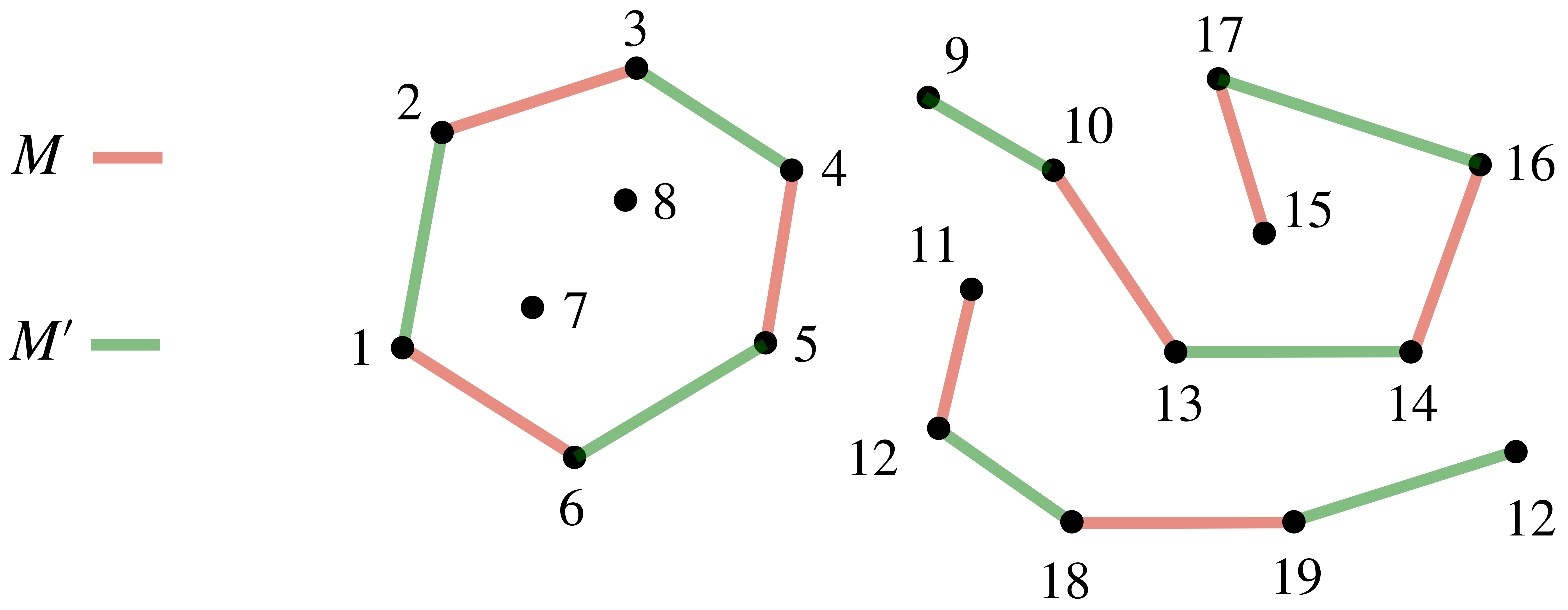
Berge's Theorem

Theorem: A matching M is maximum if and only if there is no M-augmenting path.

Proof: ($\Longleftarrow$) If there is no M-augmenting path, then M is a maximum matching.

Suppose M', not M is a maximum matching of $G = (V, E)$.

Consider the graph made of V and edges $M \oplus M'$.



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Berge's Theorem

Theorem: A matching M is maximum if and only if there is no M-augmenting path.

Proof: ($\Longleftarrow$) Connected components in $G' = (V, M \oplus M')$ can only be even length cycles or even length paths that alternate between edges of M and M'.

Therefore, $M \oplus M'$ contains equal number of edges from M and M'.

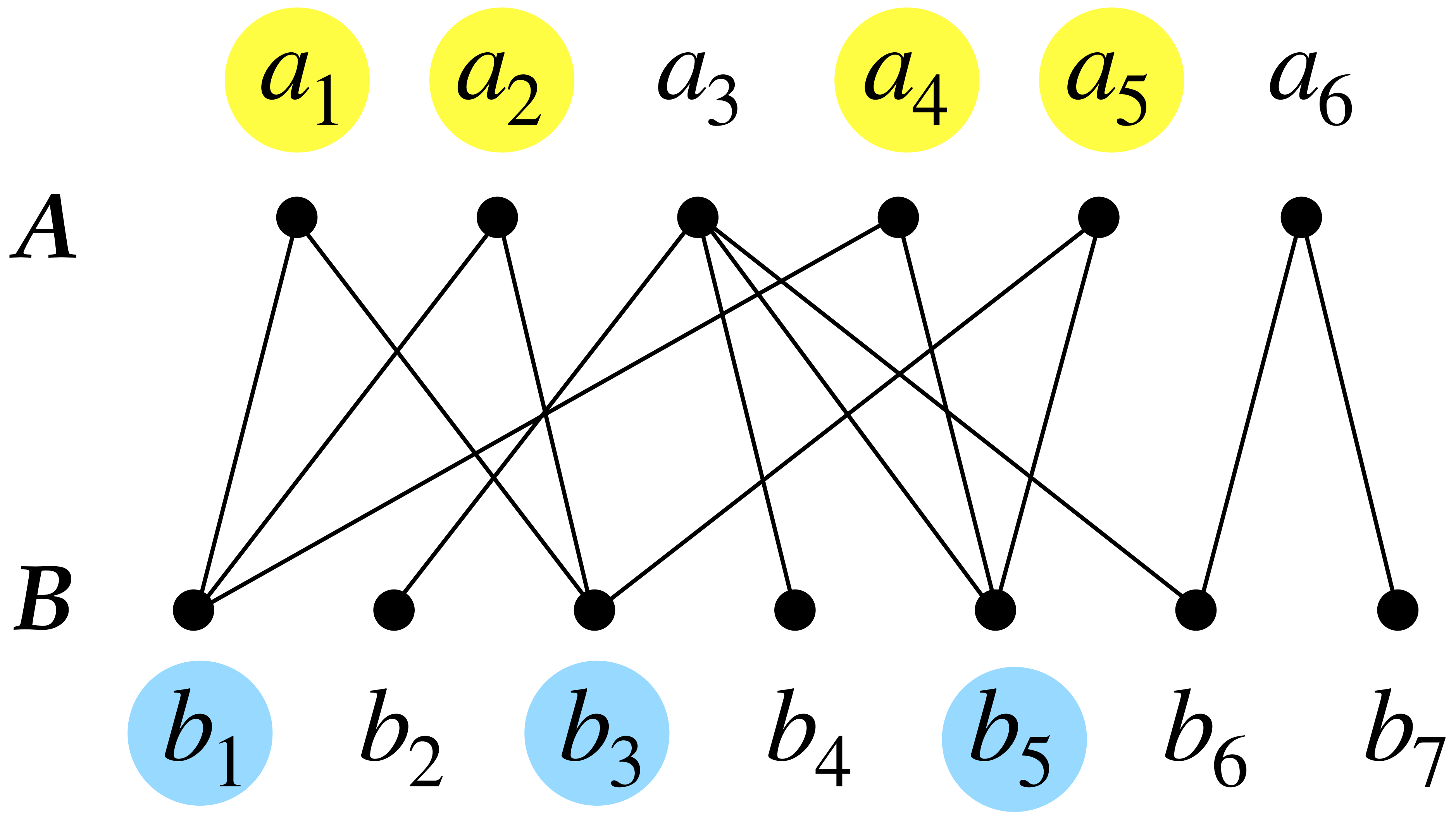
Hence, $|M| = |M'|$, a contradiction.

$\blacksquare$

Matching in Bipartite Graphs

Is there a matching in the below graph that covers A?

No, because $\{a_1, a_2, a_4, a_5\}$ has only $\{b_1, b_3, b_5\}$ as neighbours.



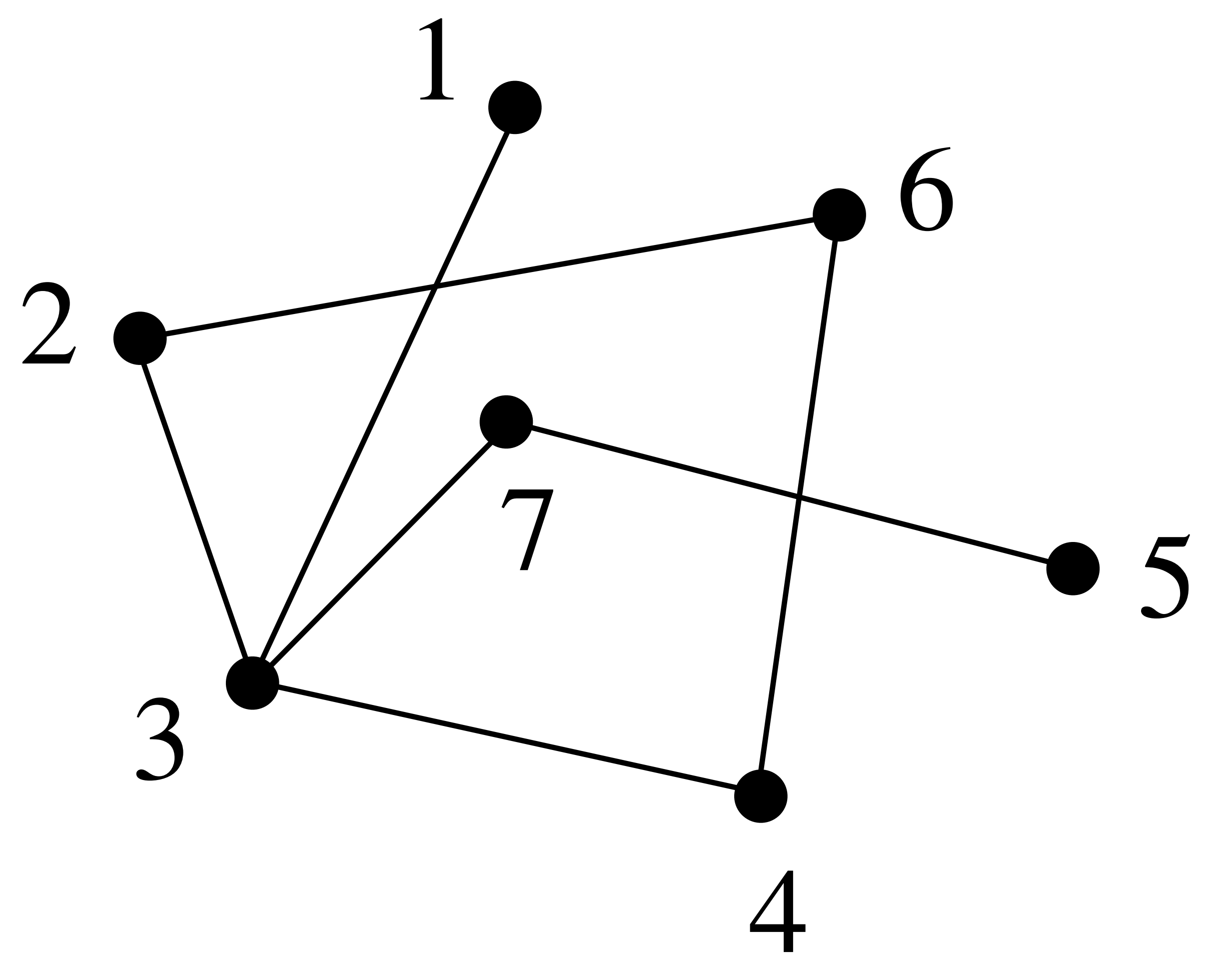
Observation: If a matching can cover A, then every subset of A has sufficient neighbours.

Neighbours of a Set of Vertices

Definition: Let $G = (V, E)$ be a graph. If $X \subseteq V$, the neighbours of X, is the set

$$N(X) = \{v \in V \backslash X \mid v \text{ is adjacent to some point in } X\}$$

Example:



$$N(\{2,3\}) = \{6,1,7,4\}$$

$$N(\{6,4\}) = \{2,3\}$$

Hall's Marriage Theorem

Theorem: Let $G = (V, E)$ be a bipartite graph with partition (A, B). G has a matching that covers A if and only if $|X| \leq |N(X)|$ for all $X \subseteq A$.

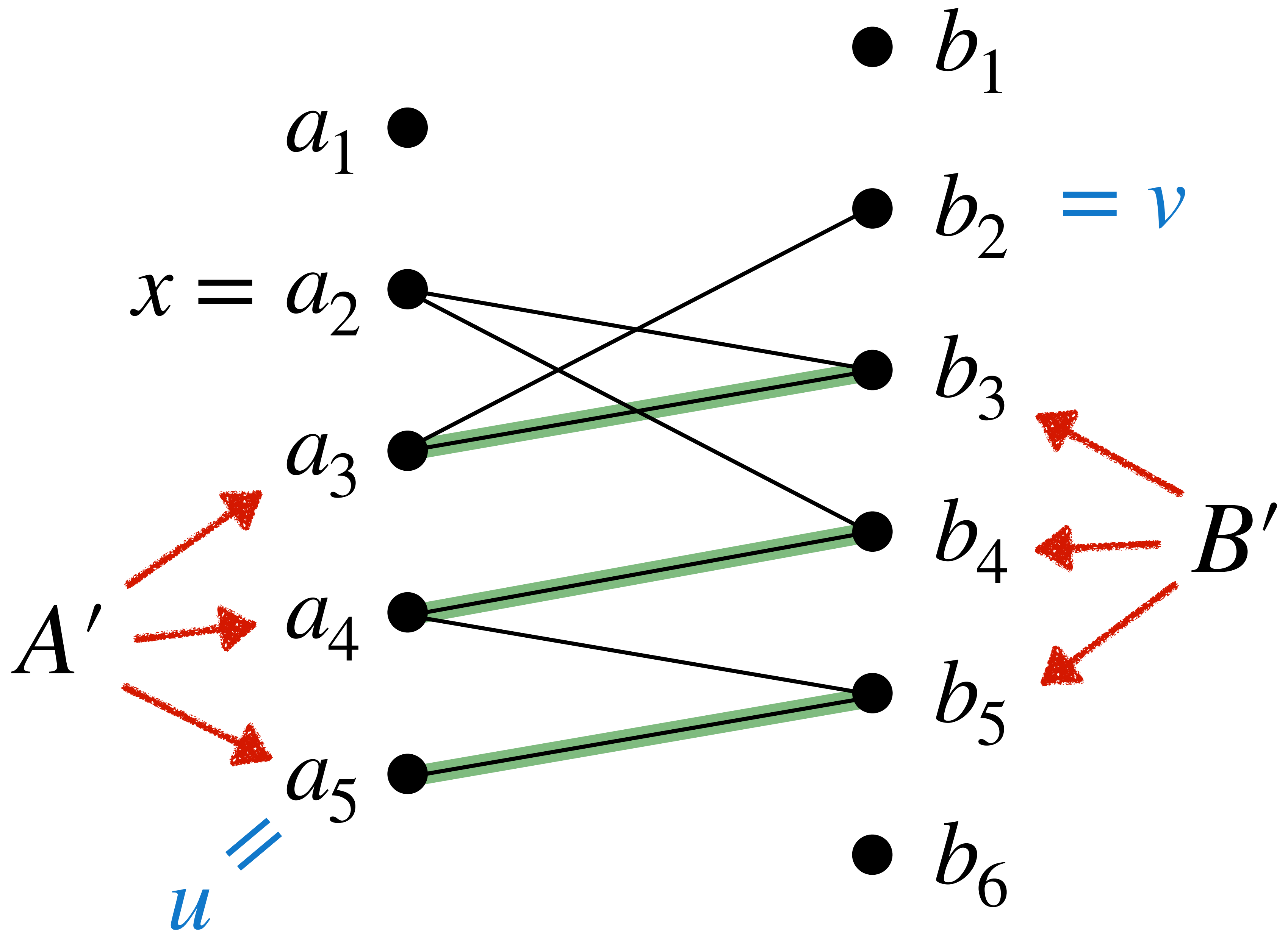
Proof: ($\Longleftarrow$) Let M be a maximum matching. We claim that M covers A.

Goal: We will show that if M does not cover a vertex $x \in A$, then there exists an M-augmenting path. Hence, a contradiction.

Let A' be the subset of A that can be reached from x using a non-trivial M-alternating path.

Let $B' \subseteq B$ be the set of penultimate vertices of such paths.

Clearly, $|A'| = |B'|$.



By marriage condition, $\exists$ an edge from a vertex u in $A' \cup \{x\}$ to a vertex v in $B \backslash B'$.

Hall's Marriage Theorem

Theorem: Let $G = (V, E)$ be a bipartite graph with partition (A, B). G has a matching that covers A if and only if $|X| \leq |N(X)|$ for all $X \subseteq A$.

Proof: $(\Longleftarrow)$ Since $u \in A' \cup x$, there must be an alternating path P from x to u.

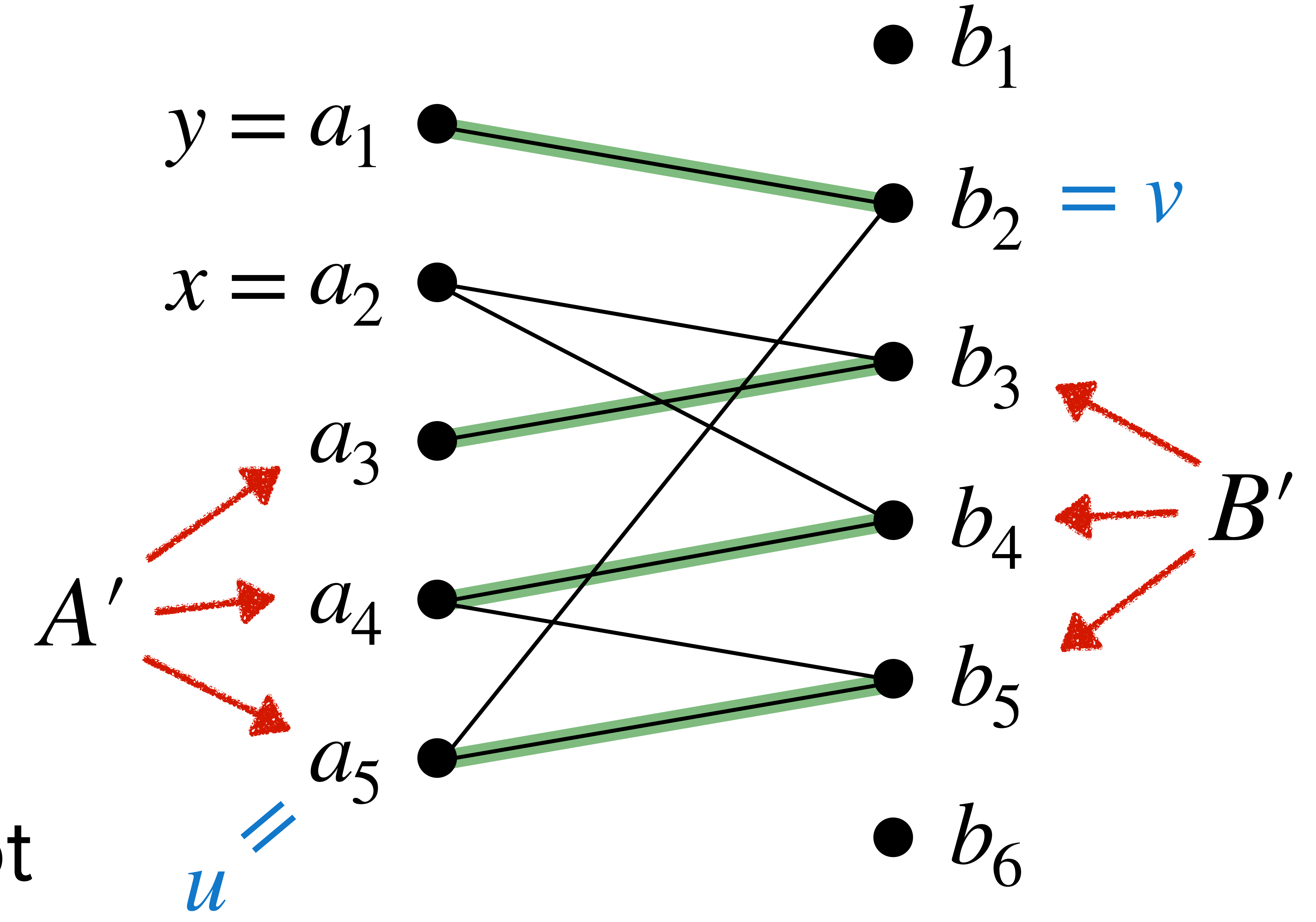
We show now that $P \,.\, \langle u, v \rangle$ is an M-augmenting path using the fact that $v \notin B'$.

- $P \,.\, \langle u, v \rangle$ is a path because ... ?
- v is uncovered:

  If v was covered by an edge $\{v, y\}$,

  then $P.\langle u, v \rangle.\langle v, y \rangle$ would be an alternating path

  from x to y, putting v in B'. A contradiction.



$P \,.\, \langle u, v \rangle$ is an M-augmenting path. Hence, M cannot not cover x.